Instrumentation and Measurement
Technology Conference - IMTC 2007
Warsaw, Poland, May 1-3, 2007

# Enhanced Time Base Jitter Compensation of Sine Waves.

Frans Verbeyst[1,2], Yves Rolain[1], Rik Pintelon[1], Johan Schoukens[1]

[1]Department ELEC, Vrije Universiteit Brussel, Pleinlaan 2, B-1050 Brussels, Belgium, yves.rolain@vub.ac.be

[2]NMDG Engineering, C. Van Kerckhovenstraat 110, B-2880 Bornem, Belgium, frans.verbeyst@nmdg.be

**Abstract -** *The goal of this paper is to estimate the amplitude of a sine wave in the presence of time base jitter, time base drift, time base distortion and additive noise. This work is motivated by a comparative study of the amplitude distortion estimated using a nose-to-nose and electro-optic sampling based calibration of a high-frequency sampling oscilloscope. It uses the exact expression of the variance of a sine wave in the presence of normally distributed additive and jitter noise, instead of a Taylor approximation of this expression.*

**Keywords -** *system identification, time base jitter, time base drift, time base distortion, high-frequency sampling oscilloscopes.*

## I. INTRODUCTION

The electro-optic sampling (EOS) system [1]-[4] at the National Institute of Standards and Technology (NIST) allows to calibrate the impulse response of an opto-electrical (O/E) converter up to 110 GHz. Using a high-frequency sampling oscilloscope, the measured impulse response corresponds to the convolution of the (known) impulse response of the O/E and that of the sampling oscilloscope. In fact, the reality is more complex because the time base drift, time base distortion and time base jitter of the oscilloscope still has to be estimated and compensated for. Also the mismatch of the O/E and the oscilloscope and the S-parameters of the adapter have to be taken into account in order to obtain the amplitude and phase distortion of the oscilloscope [5]. Having done so, it becomes possible to compare the amplitude and phase distortion of a high-frequency sampling oscilloscope plug-in, based on the nose-to-nose calibration technique [6], to the one obtained using a calibrated O/E.

A discrepancy was reported with respect to the phase distortion obtained by both calibration methods [7]. The difference starts around 20 GHz and increases as a function of the frequency. Earlier, a discrepancy was reported between the nose-to-nose based amplitude distortion and the amplitude distortion that is obtained using a stepped sine measurement [6]. The latter uses the sampling oscilloscope in freerun mode and performs a vertical histogram measurement.

This technique has some disadvantages. In freerun mode, the power measured by the scope can be derived from the standard deviation of the histogram measurement. However, the latter assumes that in freerun mode, the time axis is randomly sampled using a uniform probability density function. Also, this method does not allow to verify the presence of both harmonics and subharmonics as part of the measurement. Finally, the noise added by the sampling scope is measured without any signal being applied. Thus, it is assumed that this noise level is independent of the signal level.

As part of the verification of the reported discrepancies [6],[7], an additional method is implemented to estimate the amplitude distortion of the sampling scope using a stepped sine measurement. This method gets around the above hypotheses by using the sampling oscilloscope in triggered mode. The penalty of this method is that one has to estimate and compensate all time base errors.

This paper focuses on the required time base jitter compensation when measuring a sine wave using a high-frequency sampling oscilloscope, which also adds time base drift, time base distortion and additive noise.

In [8] it is shown that the use of a straightforward time base drift compensation incorrectly shapes the sample variance. The study also proposes higher-order models for the sample variance in the presence of both additive and jitter noise. Correct time base drift compensation in combination with a higher-order model gives very good results in the case of an impulse response measurement for realistic values of the standard deviation of both jitter and additive noise and in the case there are no model errors. Applying this technique to sine wave measurements, yields estimates of the variance of the additive noise, which strongly vary as a function of the selected model order [8] and even can become negative.

Fortunately, the exact expression for both the expected value and the variance of a pure sine wave in the presence of normally distributed jitter noise and additive noise can be derived [9],[10]. In fact, these expressions can be obtained for any distribution as long as its characteristic function is known.

## II. PROPOSED APPROACH

Consider

$$y(t_i) = A\sin\{\omega(t_i + n_t(t_i)) + \phi\} + n_y(t_i) \tag{1}$$

$y(t_i)$ represents the observation of the pure sine wave, that is contaminated by both additive noise and jitter. The noise sources are considered to be part of the observation. Both $n_y(t_i)$ and $n_t(t_i)$ are assumed to be zero mean, normally distributed with a variance of respectively $\sigma_{n_y}^2$ and $\sigma_{n_t}^2$, independent and stationary with respect to $t_i$. Both $\omega$ and $t_i$ are assumed to be known exactly.

Using the characteristic function of a normal distribution, it can be shown [9] that the expected value of $y(t_i)$ equals

$$\mu(t_i) = E\{y(t_i)\} = A \cdot e^{-\frac{\omega^2 \cdot \sigma_{n_t}^2}{2}} \cdot \sin(\omega t_i + \phi) \qquad (2)$$

Now the variance can also be calculated [10]:

$$\sigma^2(t_i) = \sigma_{n_y}^2 + \frac{A^2}{2}\left[1 - e^{-\omega^2 \cdot \sigma_{n_t}^2}\right]\left[1 + e^{-\omega^2 \cdot \sigma_{n_t}^2} \cdot \cos\{2(\omega t_i + \phi)\}\right] \qquad (3)$$

*A. Case without time base distortion*

In the absence of time base distortion, the values of $A$, $\phi$, $\sigma_{n_y}^2$ and $\sigma_{n_t}^2$ are easily retrieved using the Fourier transform of (2) and (3).

*B. Case including time base distortion*

When time base distortion is present, as it is the case during our measurements, an estimate for $A$, $\phi$, $\sigma_{n_y}^2$ and $\sigma_{n_t}^2$ can be obtained by minimizing the following cost function with respect to these unknowns:

$$V_{(W)LS} = \sum_{i=1}^{N}\left[\frac{(\mu_i - \mu(t_i))^2}{W_i^2} + \frac{(\sigma_i^2 - \sigma^2(t_i))^2}{W_i'^2}\right], \qquad (4)$$

where $\mu_i$ and $\sigma_i^2$ respectively represent the measured sample mean and sample variance at time instant $t_i$, while $\mu(t_i)$ and $\sigma^2(t_i)$ represent the corresponding models given by (2) and (3). The optional factors $W_i^2$ and $W_i'^2$ allow for a weighting. $W_i'^2$ can be based on the sample variance of $\sigma_i^2$, if available, or can be evaluated using the variance of the $\chi^2$- distribution of $\sigma_i^2$; $\sigma_i^2$ itself can be used for $W_i^2$.

*C. Classical approach*

Recent work with respect to jitter estimation [11],[12] is based on a first-order Taylor approximation

$$\tilde{y}_1(t_i) = y_0(t_i) + \left.\frac{dy_0}{dt}\right|_{t=t_i} \cdot n_t(t_i) + n_y(t_i) \qquad (5)$$

of $y(t_i) = y_0(t_i + n_t(t_i)) + n_y(t_i)$.

The variance of $\tilde{y}_1(t_i)$ equals

$$\sigma_{\tilde{y}_1}^2(t_i) = \sigma_{n_y}^2 + (dy_0/dt)^2\Big|_{t=t_i} \cdot \sigma_{n_t}^2. \qquad (6)$$

Applying this first-order approximation to (1) where $y_0(t_i) = A\sin\{\omega t_i + \phi\}$, the approximated variance equals

$$\tilde{\sigma}_1^2(t_i) = \sigma_{n_y}^2 + A^2\omega^2\{\cos(\omega t_i + \phi)\}^2 \cdot \sigma_{n_t}^2. \qquad (7)$$

Using $e^{-\delta} \cong 1 - \delta$ for $\delta \ll 1$, and only retaining first-order contributions of $\delta = \omega^2 \cdot \sigma_{n_t}^2$, (3) becomes

$$\sigma^2(t_i) \cong \sigma_{n_y}^2 + \frac{A^2}{2} \cdot \omega^2 \cdot \sigma_{n_t}^2 \cdot [1 + \cos\{2(\omega t_i + \phi)\}] \qquad (8)$$

which equals (7), because $\cos 2\varphi = 2(\cos\varphi)^2 - 1$.

Consistency with respect to the exact expression of the sample variance of a sine wave can also be shown for higher-order approximations used in [8].

*D. Maximum likelihood estimator*

Finally, a maximum likelihood (ML) estimator is proposed based on (5), which is a good approximation whenever $\omega \cdot \sigma_{n_t} \ll 1$. In that case, the sample mean of $y(t_i)$ is normally distributed, i.e. $N(\mu(t_i), \sigma^2(t_i)/n_i)$. Here, $n_i$ corresponds to the number of repeated measurements at time instant $t_i$. A model for $\mu(t_i)$ is provided by (2), while (3) can be used to model $\sigma^2(t_i)$. Furthermore, the sample variance of $y(t_i)$ is known to be $\chi^2$- distributed with $(n_i - 1)$ number of degrees of freedom. Finally, the sample mean and sample variance is known to be independent, such that the log-likelihood function results in the following cost:

$$V_{ML} = \sum_{i=1}^{N}\frac{n_i}{2} \cdot \left[\frac{(\mu_i - \mu(t_i))^2}{\sigma^2(t_i)} + \ln(\sigma^2(t_i))\right] + \sum_{i=1}^{N}\frac{(n_i - 1)}{2} \cdot \left[\frac{\sigma_i^2}{\sigma^2(t_i)}\right] \qquad (9)$$

$\mu_i$ and $\sigma_i^2$ respectively represent the measured sample mean and sample variance at time instant $t_i$. The corresponding models $\mu(t_i)$ and $\sigma^2(t_i)$ are given by (2) and (3).

## III. SIMULATIONS

First the correctness of the implementation is verified using realistic simulations.

### *A. Conditions*

Noisy realizations of a sine wave are generated according to (1), where

- $f = \dfrac{\omega}{2\pi} = 48$ GHz is known
- $t_i = i \cdot \Delta t$
- $\Delta t = 1.25$ ps, $i = 1 \,..\, N$, $N = 4000$.

$K$ sets of $K$ realizations of this sine wave are generated, with a true amplitude $A$ of 75 mV, a true phase $\phi$ of $\pi/2$ radians, zero mean normally distributed jitter noise with a standard deviation $\sigma_{n_t}$ of 1 ps and ditto additive noise with a standard deviation $\sigma_{n_y}$ of 0.5 mV. The noise sources are chosen to be independent and stationary as function of the time index $i$ and the realization index $k$. For each set of $K$ realizations, the sample mean and sample variance is calculated. Using $K$ such sets, it is also possible to calculate the sample variance of the sample variance.
In fact, simulations were performed for other levels of additive noise (1 mV and 5 mV) and are found to give similar results as the ones shown here.

### *B. Different estimators*

Results are shown for different values of $K$ (25, 100, 500), using a least squares estimator and two weighted least squares estimators based on (4) and a ML estimator based on (9). The first WLS estimator uses the sample variance of the sample variance as weighting $W_i'^2$ and is referred to as WLS1 in the tables below. The weighting $W_i'^2$ of the second WLS estimator is based on the $\chi^2$ - distribution of the actual sample variance $\sigma_i^2$ and the fact that a $\chi^2$ - distributed stochastic variable with $(n_i - 1)$ number of degrees of freedom has a mean value of $(n_i - 1)$ and a variance of $2(n_i - 1)$. Therefore $W_i'^2 = 2\sigma_i^4/(n_i - 1)$. This estimator is referred to as WLS2.

### *C. Results*

For each of these estimators, the tables below show the sample mean and the 95% confidence interval based on the sample variance of the estimated amplitude, phase, jitter variance and variance of additive noise for different data set sizes. The situations where the true value does not lie within the 95% confidence interval of the estimates are indicated in bold.

**Table 1.** Simulation results for 25 sets of 25 realizations.

| | **units** | **true value** | **LS** | **WLS1** | **WLS2** | **ML** |
|---|---|---|---|---|---|---|
| $A$ | mV | 75.00 | 75.00 ± 0.14 | **75.55 ± 0.11** | 74.97 ± 0.12 | 75.00 ± 0.08 |
| $\phi$ | rad | 1.5708 | 1.5709 ± 0.0024 | 1.5705 ± 0.0025 | 1.5702 ± 0.0057 | 1.5706 ± 0.0025 |
| $\sigma_{n_t}^2$ | $(\text{ps})^2$ | 1.000 | 1.001 ± 0.016 | **0.980 ± 0.010** | **0.810 ± 0.019** | 1.001 ± 0.012 |
| $\sigma_{n_y}^2$ | $(\text{mV})^2$ | 0.250 | 0.321 ± 3.260 | -0.584 ± 1.189 | **-7.640 ± 2.068** | 0.199 ± 1.312 |

**Table 2.** Simulation results for 100 sets of 100 realizations.

| | **units** | **true value** | **LS** | **WLS1** | **WLS2** | **ML** |
|---|---|---|---|---|---|---|
| $A$ | mV | 75.00 | 75.00 ± 0.06 | **75.14 ± 0.04** | 75.01 ± 0.04 | 75.00 ± 0.04 |
| $\phi$ | rad | 1.5708 | 1.5708 ± 0.0012 | 1.5707 ± 0.0012 | 1.5708 ± 0.0015 | 1.5707 ± 0.0014 |
| $\sigma_{n_t}^2$ | $(\text{ps})^2$ | 1.000 | 1.000 ± 0.009 | 0.996 ± 0.006 | **0.960 ± 0.008** | 1.000 ± 0.006 |
| $\sigma_{n_y}^2$ | $(\text{mV})^2$ | 0.250 | 0.299 ± 1.561 | 0.264 ± 0.763 | **-3.303 ± 0.855** | 0.236 ± 0.800 |

**Table 3.** Simulation results for 500 sets of 500 realizations.

| | **units** | **true value** | **LS** | **WLS1** | **WLS2** | **ML** |
|---|---|---|---|---|---|---|
| $A$ | mV | 75.00 | 75.00 ± 0.02 | **75.03 ± 0.02** | 75.01 ± 0.02 | 75.00 ± 0.02 |
| $\phi$ | rad | 1.5708 | 1.5708 ± 0.0005 | 1.5708 ± 0.0005 | 1.5708 ± 0.0006 | 1.5708 ± 0.0006 |
| $\sigma_{n_t}^2$ | $(\text{ps})^2$ | 1.000 | 1.000 ± 0.004 | 0.999 ± 0.003 | **0.992 ± 0.003** | 1.000 ± 0.003 |
| $\sigma_{n_y}^2$ | $(\text{mV})^2$ | 0.250 | 0.241 ± 0.667 | 0.260 ± 0.383 | **-0.600 ± 0.422** | 0.241 ± 0.420 |

It can be noticed that the LS estimator does a very good job for all considered data set sizes. However, it is outperformed by the ML estimator with respect to the uncertainty on the estimated parameters, and more specifically on the estimated amplitude and the estimated variance of the additive noise. In fact, the ML estimator outperforms all other estimators.

For both WLS estimators, a bias becomes apparent, which decreases for increasing data set sizes. For the WLS1 estimator (using the sample variance of the sample variance), the bias is clearly visible on the estimated amplitude. For small data sets, also a bias is present on the estimated variance of the jitter. For the WLS2 estimator (based on the $\chi^2$ - distribution of the sample variance), both the estimated variance of the jitter and the additive noise are biased.

It is also clear that for all estimators, the uncertainty (95% confidence interval) on the estimated variance of the additive noise is larger than the value itself, even when based on 500 realizations.

In the case of the WLS estimates, it is possible to calculate the expected value of the cost, taking into account the data set size [13] and to compare it to the actual cost. Table 4 shows that the sample mean of the actual cost of the WLS1 estimator equals the expected value within its 95% confidence interval, while the sample mean of the actual WLS2 cost is about 50% too large. As such, unfortunately it looks like the value of the WLS1 cost cannot be used to detect any anomalies, not even for small data sets.

**Table 4.** Comparison of expected and actual cost for WLS estimates.

| $K$ | expected value | $\sigma$ | actual cost WLS1 | $\sigma$ | actual cost WLS2 | $\sigma$ |
|---|---|---|---|---|---|---|
| 25 | 8723 | 151 | 8614 | 170 | 12622 | 430 |
| 100 | 8161 | 132 | 8133 | 150 | 12278 | 314 |
| 500 | 8028 | 127 | 8030 | 147 | 12292 | 297 |

Finally, after comparing the uncertainty on the estimates for $K = 500$, respectively based on their sample variance and on the corresponding element in the parameter covariance matrix, good correspondence is found for both the LS and WLS1 estimators. However, for both the WLS2 and ML estimator, using the parameter covariance matrix, the uncertainty on the estimated variance of the additive noise is underestimated by a factor of 2. Also, the sample standard deviation of the phase is 50% larger than the one indicated by the parameter covariance matrix. In the case of the estimated variance of the jitter noise, this difference reduces to 25%.

*D. Possible explanation of the bias of the WLS estimates*

For the WLS estimates, and especially with respect to WLS1, one should remember that the sample mean and the sample variance of a normally distributed stochastic variable are independent, but not for a $\chi^2$-distributed variable. Therefore, the expected value of the cost is not minimal in the true value of the parameters. The decreasing bias for larger data sets is consistent with the fact that a $\chi^2$-distribution converges to a normal distribution for an increasing number of degrees of freedom.

## IV. MEASUREMENTS

Next, the ML and LS estimators are applied to 500 repeated measurements of a sine wave at 48 GHz, using an Agilent 83480A sampling oscilloscope in combination with a 83484A 50 GHz electrical plug-in. First, a time base drift compensation is applied, as is explained in [8]. This yields the sample variance and the sample mean data on an oversampled non-equidistant time grid $t_i$. The latter is estimated based on a time base distortion measurement, which was performed up front. Also, due to the time base drift compensation [8], the number of repeated realizations $n_i$ varies as a function of the time grid index $i$.

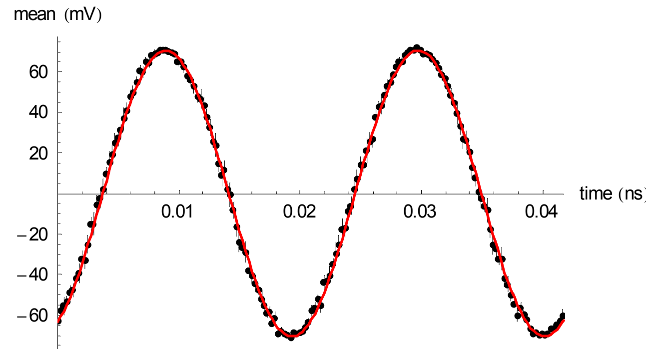


**Figure 1.** First two periods of the measured sample mean at 48 GHz (black dots), its 95% confidence interval (vertical black lines) and the estimated mean (red solid line).

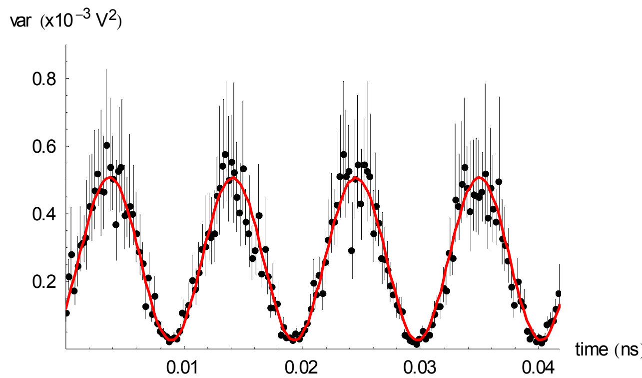


**Figure 2.** Corresponding measured sample variance at 48 GHz (black dots), its 95% confidence interval (vertical black lines) and the estimated variance (red solid line).

Figure 1 and figure 2 show the sample mean and sample variance of the first two periods of the sine wave measurement obtained at an excitation frequency of 48 GHz for the ML estimator. The subharmonic at 24 GHz and the second harmonic at 96 GHz, which are neglected by the model, are found to be more than 45 dB down with respect to the fundamental.

The estimated mean and variance based on the LS estimator cannot be distinguished from those based on the ML estimator.

Although the estimator also provides estimates for $\phi$, $\sigma_{n_y}^2$ and $\sigma_{n_t}^2$, the main parameter of interest here is $A$. It corresponds to the amplitude of the sine wave "before" the low-pass effect of the jitter. The LS estimator yields an amplitude of 74.16 mV ± 0.03 mV (95% confidence interval), while its ML equivalent provides a value of 74.12 mV ± 0.02 mV (95% confidence interval). Both estimates match within their 95% confidence interval, while the uncertainty of the ML estimate is about 50% smaller than that of the LS estimate.

Table 5 allows to compare the other estimated parameters in more detail. All estimates match within their 95% confidence intervals, which are based on the parameter covariance matrix. One should keep in mind that the simulations showed that the parameter covariance matrix of the ML estimator underestimates the uncertainty, mainly on the estimated variance of the additive noise and on the phase.

**Table 5.** LS and ML estimates for the sine wave measurement at 48 GHz.

| | units | LS | 95% | ML | 95% |
|---|---|---|---|---|---|
| $A$ | mV | 74.16 | 0.03 | 74.12 | 0.02 |
| $\phi$ | rad | -1.1155 | 0.0006 | -1.1166 | 0.0005 |
| $\sigma_{n_t}$ | ps | 1.056 | 0.002 | 1.057 | 0.002 |
| $\sigma_{n_y}$ | mV | 1.281 | 0.326 | 1.650 | 0.083 |

## V. CONCLUSIONS

This article estimates the amplitude of a sine wave in the presence of time base jitter, time base drift, time base distortion and additive noise. It uses the exact expression of both the expected value and the variance of a sine wave in the presence of normally distributed additive and jitter noise. Realistic simulations show that the LS estimator does a very good job for different data set sizes. The simulations also show that the ML estimator is to be preferred due to its efficiency, especially for limited data sets. For these data set sizes, the WLS estimators should be used with care. Finally, the estimators are applied to measurements, which were performed to verify the discrepancies reported by [6] and [7]. The 95% confidence interval for the estimated amplitude at 48 GHz is well within the 0.1 dB required by the additional method, which was implemented to estimate the amplitude distortion of a high-frequency sampling oscilloscope using a stepped sine measurement.

## VI. ACKNOWLEDGEMENT

The authors thank Paul Hale of NIST for pointing out that (3) was already derived by J. W. Chapman [10].

## VII. REFERENCES


[1] D. Williams, P. Hale, T. Clement, and J. Morgan, "Mismatch corrections for electro-optic sampling systems," 56th ARFTG Conference Digest, pp. 141-145, Nov. 30-Dec. 1, 2000.

[2] D. Williams, P. Hale, T. Clement, and J. Morgan, "Calibrating electro-optic sampling systems," Int. Microwave Symposium Digest, Phoenix, AZ, pp. 1527-1530, May 20-25, 2001.

[3] T. Clement, D. Williams, P. Hale, and J. Morgan, "Calibrating photoreceiver response to 110 GHz," Proc. 15th Annual Meeting, IEEE Lasers and Electro-optics Soc., Glasgow, Scotland, 2002.

[4] D. Williams, P. Hale, T. Clement, C-M. Wang, "Uncertainty of the NIST Electro-optic Sampling System," NIST Technical Note 1535, 2004.

[5] T. Clement, P. Hale, D. Williams, C. Wang, A. Dienstfrey, and D. Keenan, “Calibration of Sampling Oscilloscopes With High-Speed Photodiodes,” IEEE Transactions on Microwave Theory and Techniques, Vol. 54, No. 8, pp. 3173-3181, August 2006.

[6] P. Hale, T. Clement, K. Coakley, C. Wang, D. DeGroot and A. Verdoni, “Estimating the Magnitude and Phase Response of a 50 GHz Sampling Oscilloscope Using the “Nose-To-Nose” Method,”, 55th ARFTG Conf. Digest, June 2000.

[7] D. Williams, P. Hale, T. Clement, “Electrical-phase Traceability to NIST’s EOS System,” research update presented at the 4th ARFTG NVNA User’s Forum, June 2004 (http://www.arftg.org/LSNA/4th/UsersForum_June2004_Minutes2.pdf)

[8] F. Verbeyst, Y. Rolain, J. Schoukens, R. Pintelon, “System Identification Approach Applied to Jitter Estimation”, IMTC Conference Proceedings, pp. 1752-1757, winner of a "Honorable mention recognized by the Award Commission of Agilent Technologies", IMTC ‘06, April 2006, Sorrento, Italy.

[9] T. Souders, D. Flach, C. Hagwood and G. Yang, “The Effects of Timing Jitter in Sampling Systems,” IEEE Transactions on Instrumentation and Measurement, Vol. 39, No. 1, pp. 80-85, February 1990.

[10] J. W. Chapman, “Moments, variances, and covariances of sines and cosines of arguments which are subject to random error,” Technometrics, Vol. 12, pp. 693-694, 1970.

[11] G. Vandersteen and R. Pintelon, “Maximum Likelihood Estimator for Jitter Noise Models,” *IEEE Transactions on Instrumentation and Measurement*, Vol. 49, No. 6, December 2000.

[12] K. Coakley, C.-M. Wang, P. Hale and T. Clement, “Adaptive Characterization of Jitter Noise in Sampled High-Speed Signals,” *IEEE Transactions on Instrumentation and Measurement*, Vol. 52, No. 5, October 2003.

[13] J. Schoukens, R. Pintelon and Y. Rolain, “Maximum Likelihood Estimation of Errors-In-Variables Models using the Sample Covariance Matrix Obtained from Small Data Sets,” published as part of “Recent Advances in Total Least Squares Techniques and Errors-In-Variables Modeling”, Sabine Van Huffel (editor), Siam, Philadelphia, 1997.