Short Wire Antennas: A Simplified Approach
Part I: Scaling Arguments

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0. Introduction:

How does a wire dipole antenna work? How do we find the resistance and the reactance? Why does the reactance vanish at an appropriate length or frequency?

In typical textbook treatments [1-3] these problems are approached indirectly: The resistance is estimated by calculating the fields from a known current distribution along the wire at infinite distance, finding the consequent radiated power by integrating the Poynting vector over an arbitrarily large sphere, and setting the result equal to the product I^2R to find the equivalent resistance ('radiation resistance'). The reactance is calculated by solving Pocklington's or Hallen's integral equation numerically (the finite-element solution to this problem is generally referred to as the Method of Moments), or by integrating the Poynting vector over the antenna surface and equating it to the delivered power (induced emf). These approaches are of course perfectly correct, but perhaps more obscure than they need to be. In this article we shall try to illustrate a simpler and more direct way of understanding how short wire antennas, and by extension other small antennas, interact with traveling electromagnetic waves, in which we focus on the potentials that result directly from charges and currents.

We shall use only three pieces of basic physics:

- Time-delayed Coulomb's law: each element of charge contributes an electric potential ϕ inversely proportional to the distance from the point of measurement, where the charge is evaluated at an earlier time corresponding to the propagation delay:

$$\phi(r,t) = \frac{\mu_0 c^2}{4\pi} \iiint \frac{q\left(r', t - \frac{|r'-r|}{c}\right)}{|r'-r|} dv \qquad (1.1)$$

- Time-delayed Ampere's law: each element of current contributes a vector potential **A** inversely proportional to the distance, and time-delayed. The vector potential **A** is oriented in the same direction as the current.

$$\mathbf{A}(r,t)=\frac{\mu_0}{4\pi}\iiint\frac{J\left(r',t-\frac{|r'-r|}{c}\right)}{|r'-r|}dv \qquad (1.2)$$

- The resulting electric field is the sum of the potential gradient and the time derivative of the vector potential:

$$\mathbf{E}=-\frac{\partial}{dt}\mathbf{A}-\nabla\phi=-i\omega\mathbf{A}-\nabla\phi \qquad (1.3)$$

where the second step assumes harmonic time dependence. The voltage from one point to another is the line integral of the electric field. Note that no explicit use of the magnetic field is required; we can banish cross-products and curls from consideration.

1. A Wire in Space

Consider a wire suspended in space, with an impinging vector potential **A** and consequent electric field $-i\omega\mathbf{A}$ (figure 1). For reasonable wire thicknesses, we can assume that currents and charges are only present in a thin layer on the surface of the wire, and adjust themselves to ensure zero field well within the wire. How is this arranged?

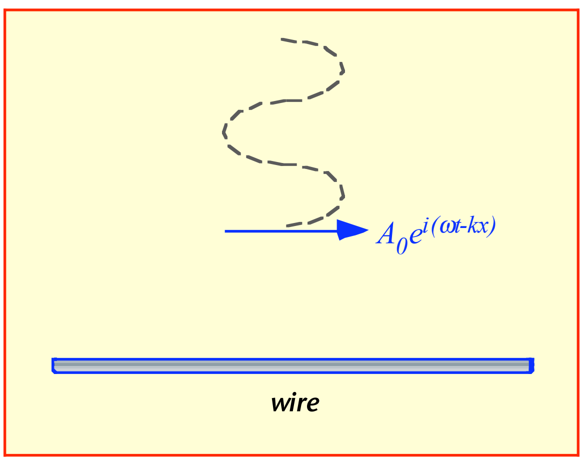

Figure 1: wire and impinging potential; for simplicity, ***A*** *is taken along the axis of the wire and the direction of propagation is perpendicular to the wire.*

To analyze the situation we begin with the basic source relationships: each infinitesimal volume of charge creates an electrostatic potential ϕ, decreasing inversely with distance and traveling at the speed of light. Similarly, each infinitesimal current element creates a vector potential **A**, oriented in the direction of the current, also decreasing inversely with distance and propagating at the speed of light.

The key to relating the current to the incident electric field is to decompose the ***scattered*** potential that arises due to current flow into three components, each of which has a distinct physical origin and differing dependence on the geometry of the wire:

- An instantaneous electric potential ϕ_{sc} results from the accumulation of charge, primarily near the ends of the wire, where charge must build up since current cannot flow past the ends. For short antennas we can ignore the time delay between the charge location *r'* and the axis of the wire *r*.
- An instantaneous magnetic component $A_{sc,in}$, in phase with the local current, whose value at each location is mainly determined by the current in that vicinity.
- A delayed magnetic component $A_{sc,d}$, dependent on the integral of the total current along the wire. For harmonic time dependence, this component lags the instantaneous component by 90 degrees – that is, it is along the negative imaginary axis (figure 2).

Corresponding to each scattered potential is an induced voltage. We shall somewhat arbitrarily choose to measure this voltage from left to right; the electrostatic voltage is thus the negative of the absolute potential, and the magnetic components are the line integral of the corresponding electric field along the wire. These three contributions are combined to obtain the net ***scattered voltage***; we then arrange the phase and amplitude of the current so that this scattered voltage cancels the incident voltage, ensuring that the interior of the wire is field-free as it must be.

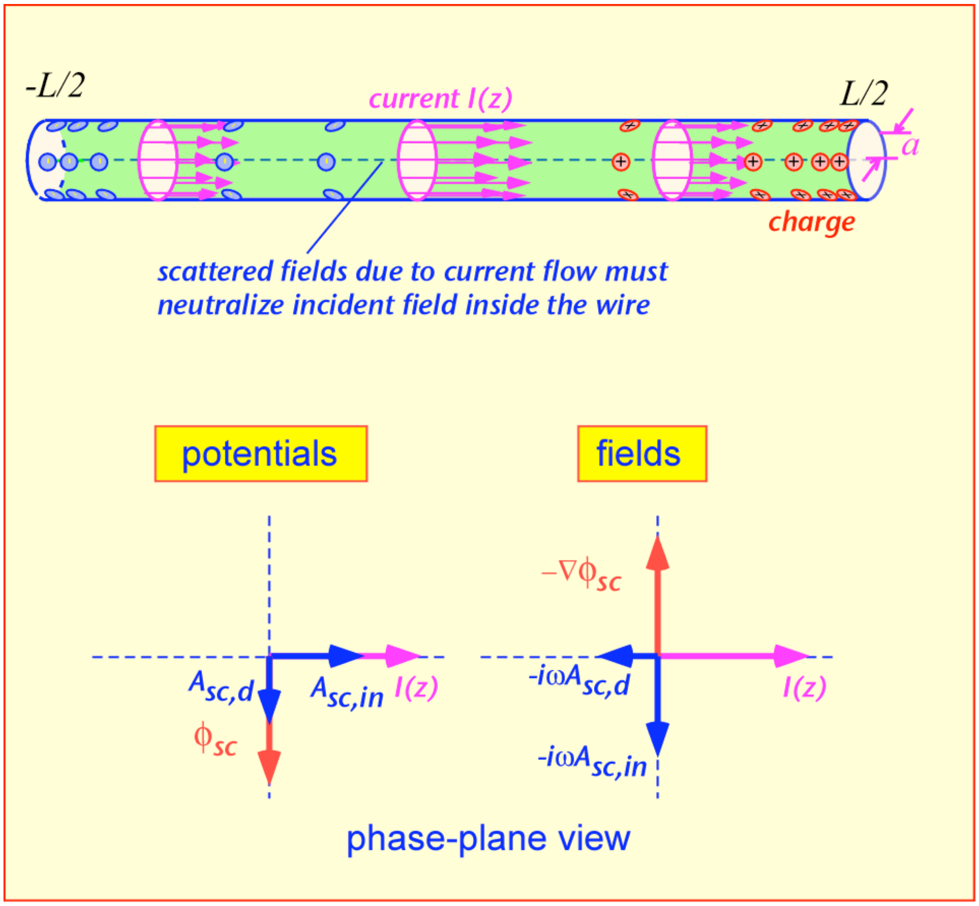

Figure 2: current and charge on the wire; resulting potentials and fields and their phase relationship with the current

Let's examine in a general way how each of these scattered components depends on the geometry of the wire. The total charge induced on each half of the wire is the integral of the current, from the simple relation that the current I is the charge Q divided by the time t:

$$I = \frac{Q}{t} \tag{1.4}$$

Thus by Coulomb's law, the potential is roughly (figure 3):

$$\phi_{sc} \propto \mu_0 c^2 \frac{Q}{L} \tag{1.5}$$

For harmonic time dependence:

$$I \propto e^{i\omega t} \rightarrow Q \propto \frac{I}{i\omega} \tag{1.6}$$

so the potential scales as:

$$\phi_{sc} \propto -i\mu_0 c^2 \frac{I_0}{i\omega L} \tag{1.7}$$

Using the relation $\omega = \frac{2\pi c}{\lambda}$, we have:

$$V_{sc,\phi} \propto i\mu_0 c I_0 \left(\frac{\lambda}{L}\right) = iV_0 \left(\frac{\lambda}{L}\right) \tag{1.8}$$

where $V_0 = \mu_0 c I_0$, the product $\mu_0 c$ being the impedance of free space, 377 Ω, so that V_0 is a sort of characteristic voltage associated with the current I_0.

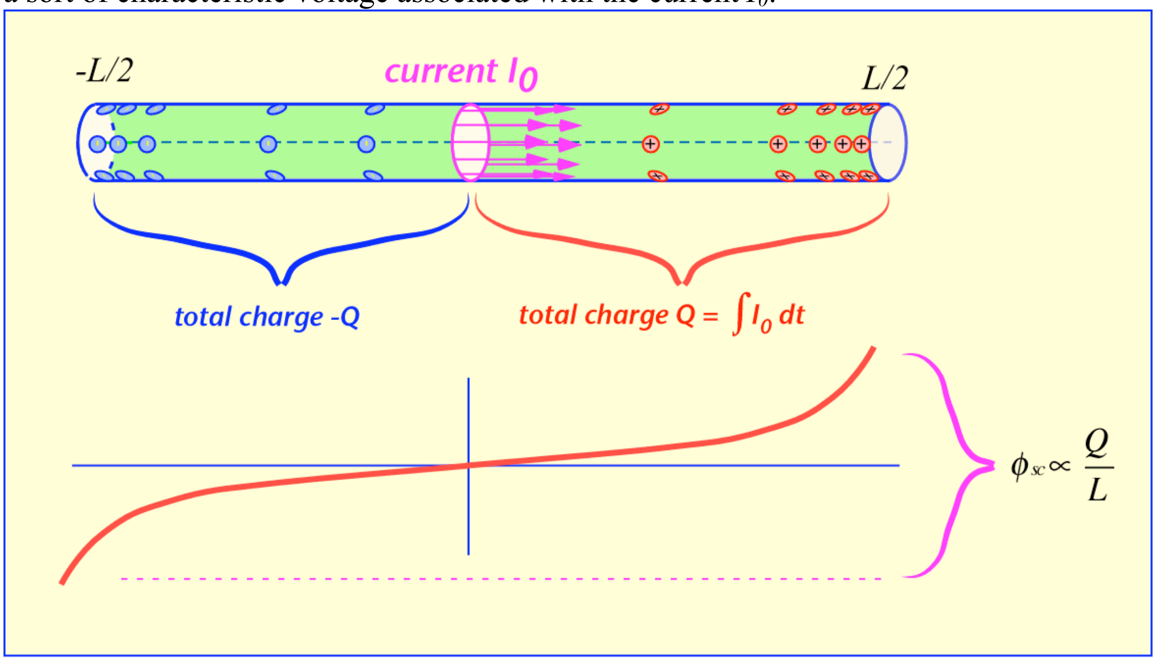

Figure 3: schematic depiction of electrostatic potential along the center of the wire, shown delayed by 1/4 cycle from the current

Physically this equation tells us that at lower frequencies the charge has a longer time to accumulate and thus grows larger in magnitude, and the resulting field is larger when the wire is short and the charges are close together. Thus the electrostatic potential is important at low frequencies and short wire lengths.

The ‘instantaneous’ or magnetostatic vector potential is taken to be the sum of contributions from all the current elements weighted by (1/distance) from the point of measurement, with any propagation delay ignored. We will choose to find the potential along the axis of the wire, which greatly simplifies the calculation. From the potential version of Ampere’s law we have:

$$A_{sc,\,in} = \frac{\mu_0}{4\pi}\iiint \frac{J}{r}\,dv \approx \frac{\mu_0}{4\pi}\int \frac{I}{r}\,dz \tag{1.9}$$

where the second expression arises from assuming that the currents are localized on the surface of the wire, and uniform around the wire, so that all the current elements at an axial position *z* are at the same distance from the point of interest. The largest contribution to the integral arises from nearby currents (figure 4), so the potential at any given location along the wire is in phase with and proportional to the current flow, but only weakly dependent on the length of the wire (we shall find the dependence to be logarithmic).

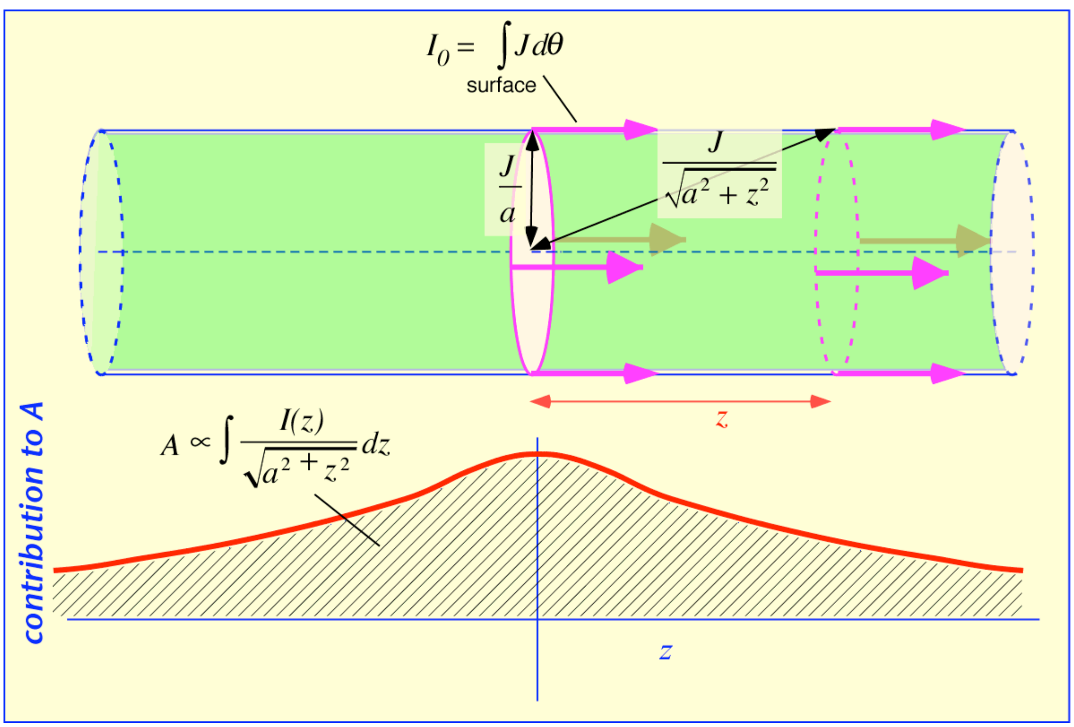

Figure 4: contributions to the instantaneous vector potential along the axis of the wire

The electric field due to this potential thus scales roughly as:

$$E_{sc,\,in} \propto -i\omega\mu_0 I_0 \propto -i\mu_0 c I_0 \frac{1}{\lambda} \tag{1.10}$$

The resulting voltage on the wire is roughly the product of the field and the length:

$$V_{sc,\,in} \propto -i\mu_0 c I_0 \left(\frac{L}{\lambda}\right) = -iV_0 \left(\frac{L}{\lambda}\right) \tag{1.11}$$

where we have employed the characteristic voltage V_0 defined in equation (1.5). By comparison of equations (1.5) and (1.8), we see that the voltages due to the instantaneous charges and current are of opposite sign and scale inversely with respect to the normalized length of the antenna. If, for example, we start near DC and increase the frequency of the impinging radiation, the electrostatic contribution will fall and the magnetostatic contribution will rise (figure 5). It is reasonable to guess, as depicted in the figure, that at some frequency the two might cancel: that is, a ***resonant*** frequency is likely to exist. At resonance, the current generates no net voltage along the wire (in this approximation); in order to cancel the incident voltage and create zero electric field in the wire, we would require an infinite amount of current to flow. In practice, the current does grow large at resonance, but is limited by the effects of the delayed potential, as we shall show in a moment.

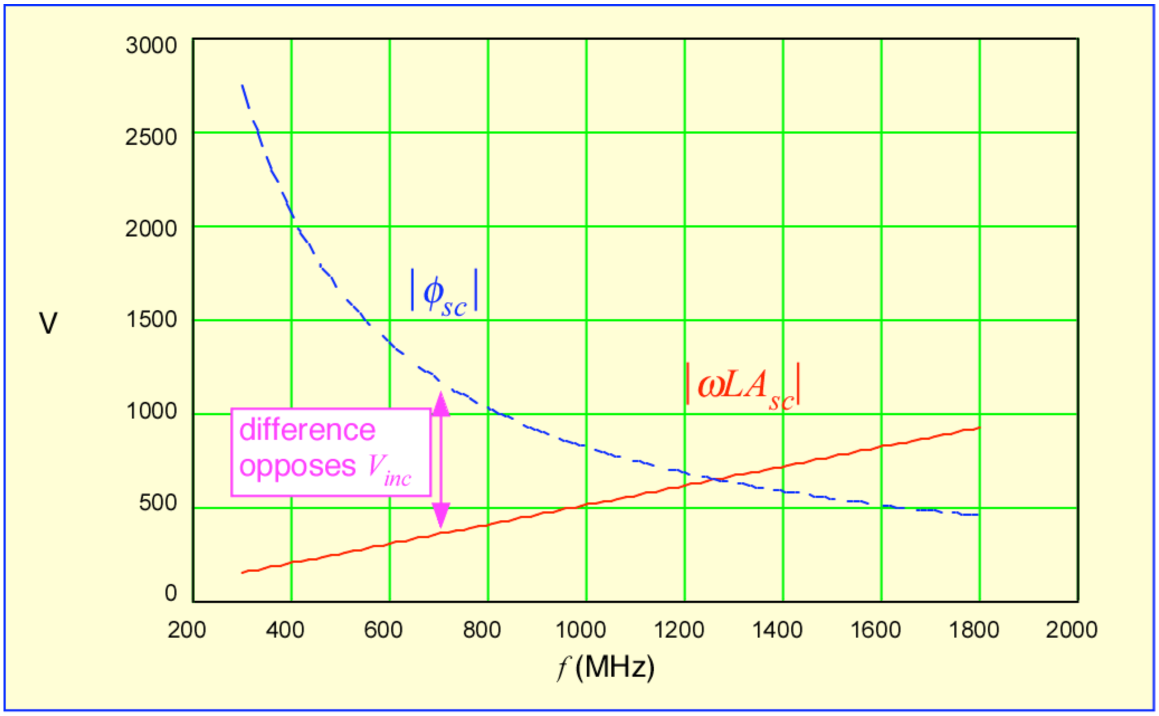

Figure 5: example of electrostatic and magnetostatic contributions to the scattered voltage as a function of frequency; current of 1 ampere, antenna length 0.1 m

Finally, the delayed component is the result of considering the finite speed of light. The potential due to currents from far away is increasingly delayed – for harmonic time dependence, more of that potential is along the delayed (negative imaginary) axis. To first order, this effect cancels the (1/r) decrease in the contribution of more distant currents, so that the contribution of each current element to the delayed potential is independent of position (assuming, of course, that the antenna is short compared to a wavelength). Mathematically, for a harmonic time dependence, the contribution to the potential of any current element must be multiplied by an exponential term e^{-ikr}. When we expand the exponential to first order, (1-*ikr*), we find that we have already accounted for the first term, which is the instantaneous potential. The first-order delayed term, -*ikr*, is linearly dependent on the distance between the current element and the point of measurement; this linear dependence compensates for the inverse weighting of the more distant currents to give a potential integral with no dependence on the distance between the measurement and the currents:

$$A_{sc,d} = \frac{\mu_0}{4\pi}\int \frac{J(-ikr)}{r}dv = -ik\frac{\mu_0}{4\pi}\int Idz \qquad (1.12)$$

The delayed component is thus linearly proportional to the wavevector $k = 2\pi/\lambda$. If the current doesn't vary too much over the wire, the integral is roughly just the product of the current and the wire length (figure 6). We obtain:

$$A_{sc,\ d} \propto -ik\mu_0 I_0 L \propto -i\mu_0 I_0 \left(\frac{L}{\lambda}\right) \tag{1.13}$$

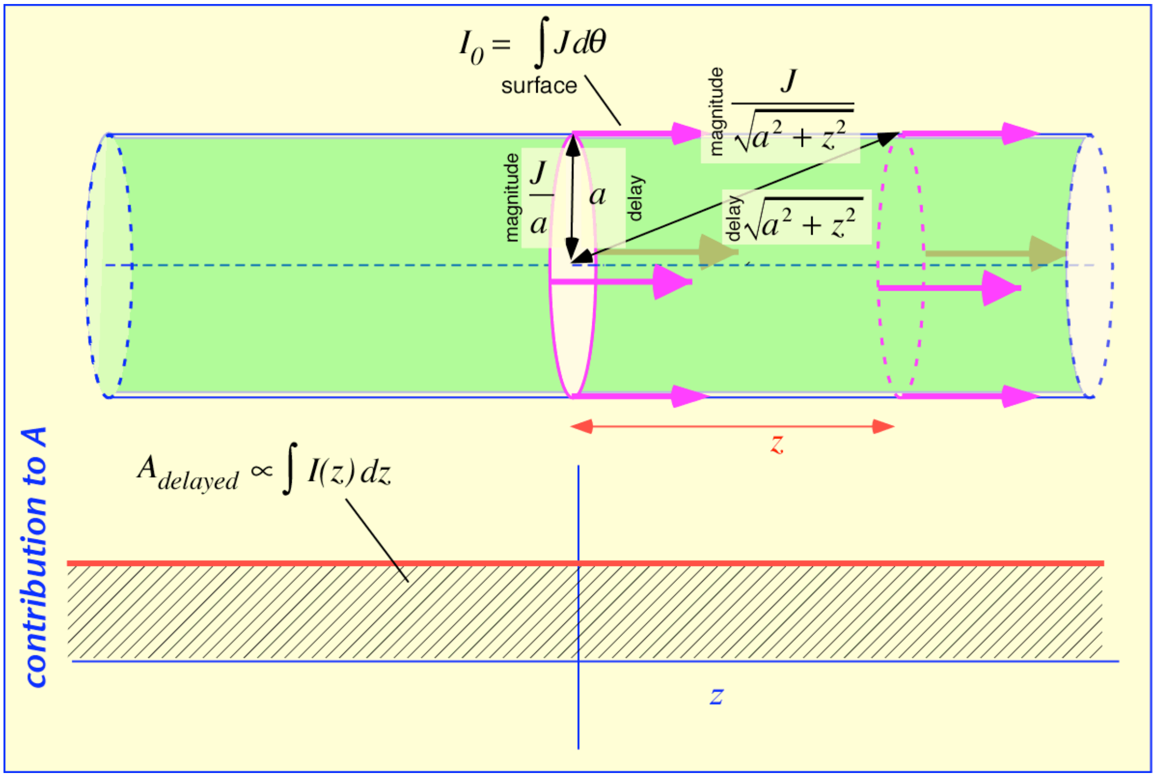

Figure 6: contributions to delayed vector potential along the axis of the wire; as long as the delay is small increasing delay compensated decreasing magnitude, so that all parts of the wire contribute equally.

The induced electric field scales as

$$E_{sc,\,d} \propto -i\omega A_{sc,\ d} \propto -\mu_0 c I_0 \left(\frac{L}{\lambda^2}\right) \tag{1.14}$$

and the induced voltage is once again of the product of the field and the wire length:

$$V_{sc,\,d} \propto -V_0 \left(\frac{L}{\lambda}\right)^2 \tag{1.15}$$

Note that the electric field in equation (1.11) is real and opposed to the current. If this is the only contribution to the field, in order to cancel the incident field the current will be in phase with the incident field so that the scattered and incident fields are of opposite sign: that is, the current is flowing in phase with the incident field. The wire is ***acting like a resistor*** even though we have completely ignored the finite conductivity of the wire. We can write $I = V_{inc}/R_{rad}$, where R_{rad} is known as the ***radiation resistance*** of the wire. The energy dissipated by this apparent resistance is radiated away to the distant world, though the value of the resistance is obtained through a purely local calculation[1].

Since the total scattered voltage must be equal in magnitude to the incident voltage, we can write a simple expression for the current:

$$I_0 \propto \frac{\left|V_{inc}\right|}{\mu_0 c \sqrt{\left(\kappa_1\left(\frac{\lambda}{L}\right) - \kappa_2\left(\frac{L}{\lambda}\right)\right)^2 + \kappa_3\left(\frac{L}{\lambda}\right)^4}};$$

$$\text{at resonance } I_0 \propto \frac{\left|V_{inc}\right|}{\mu_0 c} \tag{1.16}$$

where the κ's are as-yet-undetermined constants of order 1. Figure 7 summarizes the relationship between incident voltage, current, and scattered voltage, for various possible values of the scaling parameter (L/λ).

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[1] In fact, we have made an important assumption about the outside world, in asserting that only retarded potentials are present. A time-symmetric electrodynamics can be formulated by abandoning this assumption, at the cost of making thermodynamic assertions about the distant universe; see Mead [4] and references therein.

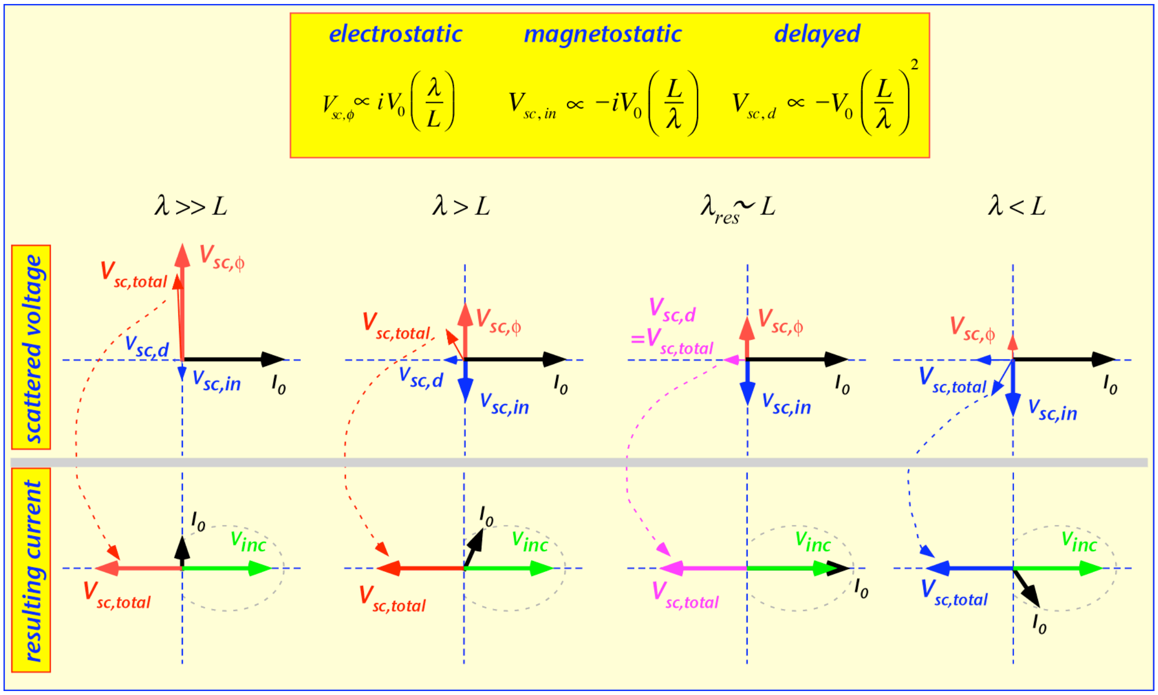

Figure 7: phase and amplitude of scattered voltages relative to induced current (top row) and incident voltage (bottom row) for various values of normalized length

At long wavelengths (low frequencies) the scattered voltage is dominated by the strong electrostatic contribution. The current must lead the incident voltage by 90 degrees, but only a small current is needed. With increasing frequency, wavelength becomes only modestly longer than the wire. The electrostatic voltage contribution is partially cancelled by the magnetostatic voltage, and the delayed component becomes significant; the overall phase of the scattered voltage is larger, and thus the current moves closer to the real axis. At resonance, the electrostatic and magnetostatic contributions cancel, leaving only the small delayed voltage. A large current, in phase with the incident voltage, is required to cancel the incident voltage. At still higher frequencies and smaller wavelengths, the magnetostatic contribution begins to dominate the scattered voltage, and the current begins to lag the voltage. We can only proceed a short ways past resonance before more sophisticated approximations are needed, as the wire is no longer short compared to a wavelength. Note that in this discussion we have suppressed all the constant terms in the interests of simplicity. In fact, we will find that resonance occurs when the wavelength is a bit more than twice the length of the wire.

So far we have examined the scattered potentials and fields along the axis of the wire. However, the source equations apply everywhere. At large distances from the wire, the current on the antenna gives rise to a potential delayed by the speed of light; for a harmonic disturbance the delay simply introduces a phase shift. Perpendicular to the axis of the wire, the distance between a test point and any point on the wire is equal when the distance is large. Thus the vector potential is simply the integral of current over the length of the wire – the same integral that determines the delayed constituent of the local magnetic potential – divided by the distance (equation 1.16, figure 8):

$$A_{sc,\ far} \propto \mu_0 \frac{e^{-ikr}}{r} \int I dz \approx \mu_0 \frac{e^{-ikr}}{r} I_0 L \qquad (1.17)$$

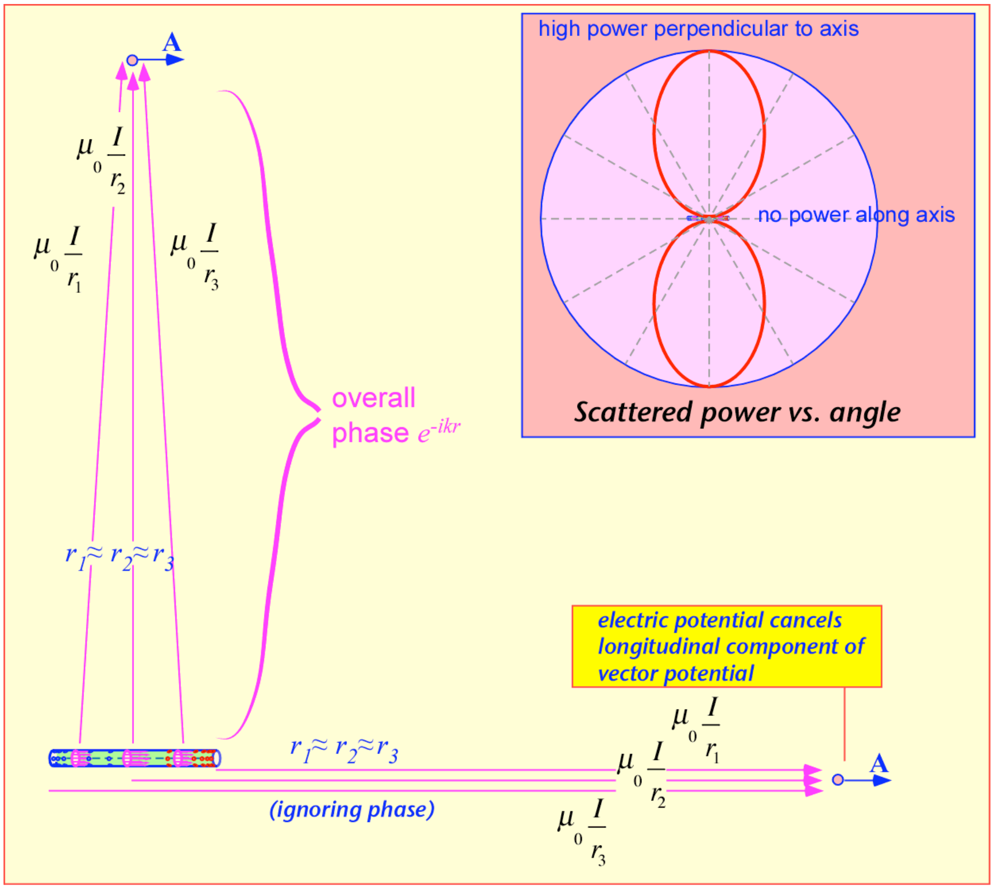

Figure 8: scattered potential at long distances is the sum of contributions from current along the wire; only the transverse component (perpendicular to r) contributes to net power transfer

Since the current is purely along the axis of the wire, the vector potential everywhere is directed parallel to the wire; but it can be shown that only that part of the potential perpendicular to the direction of propagation couples with distant charges or currents, the longitudinal component being cancelled by the electric potential contribution [5]. Projection of the potential perpendicular to the vector **r** introduces a factor of sin(θ) where θ is the angle between **r** and the axis of the wire. (A complete calculation would obtain a more complex angular dependence, due to the variation in distance and thus phase for current elements at differing locations on the wire, but the distinction is of minor import for short antennas.) The wire scatters part of the incident energy, predominantly in the plane perpendicular to the wire axis (figure 8, inset). Since the amount of scattering is proportional to the current, the scattering cross-section is maximized at the resonant frequency.

We can also estimate the total scattered power from the wire. The characteristic voltage V_0 is the product of the current and the impedance of free space, $\mu_0 c$ = 377 Ω, so the current at resonance is of the order of the incident voltage divided by $\mu_0 c$ (equation 1.16).This current in turn produces a scattered potential in the far field:

$$A_{sc} \propto \frac{\mu_0 I_0 L}{r} = \frac{\mu_0 L}{r}\frac{V_{inc}}{\mu_0 c} = \frac{L}{r}\frac{V_{inc}}{c} \approx \frac{L}{r}\frac{\omega A_{inc} L}{c} \tag{1.18}$$

The radiated power density has the form of the square of the local electric field divided by the impedance of free space (that is, V^2/R per unit area):

$$U \propto \frac{|\omega A|^2}{\mu_0 c} = \frac{\omega^2}{\mu_0 c}\left(\frac{\omega A_{inc} L^2}{rc}\right)^2 = \underbrace{\frac{\omega^2 A_{inc}{}^2 L^2}{\mu_0 c}}_{P_{inc}}\frac{\omega^2 L^2}{r^2 c^2} \tag{1.19}$$

Note that we have written the radiated power density in terms of a quantity P_{inc}, which is simply the incident power on a square region L on a side. Ignoring the angular dependence (which introduces a constant factor of order 1) the total radiated power scales as:

$$P \propto Ur^2 = \underbrace{\frac{\omega^2 A_{inc}{}^2 L^2}{\mu_0 c}}_{P_{inc}}\frac{\omega^2 L^2}{c^2} \propto P_{inc}\left(\frac{L}{\lambda}\right)^2 \approx P_{inc} \text{ at resonance} \tag{1.20}$$

That is, to within a constant factor, the amount of power scattered by a wire is equal to the power that falls on a square region of are about L^2: the scattering cross-section of the wire is quite large even if the wire is extremely thin.

Let's get a feeling for the size of the various quantities we've introduced. Remember that these are only order-of-magnitude values since we have not yet attempted to derive the constant terms and parameter dependencies. An impinging voltage of 1 V produces a current on the order of (1/377) ≈ 2.5 mA, for a resonant antenna on the order of a wavelength long.). If the length of the antenna is (say) 0.1 meter, the electric field is about 10 V/m and the scattered power is around (100/377)(0.01) or about 3 mW. A short wire (say on the order of $\lambda/10$) will have a capacitive current around (1/10) of this value, and a real current (in phase with the impinging field) of perhaps (1/100).

Note that since the real current scales as the second power of the wavelength, the scattered power (which goes as the square of the current) scales as the ***fourth power of the ratio of antenna size to wavelength***. This may be a familiar result: it is known as ***Rayleigh scattering***, and with antennas the size of individual molecules explains why the cloudless sky is blue.

2. Wires and Antennas

In order to use a wire as a means to convert between voltages and waves – as an ***antenna*** – we need to have a method of making a connection. One common approach is to break the wire in the center and connect one lead of a balanced transmission line to each end: a ***dipole antenna***, as depicted in figure 9.

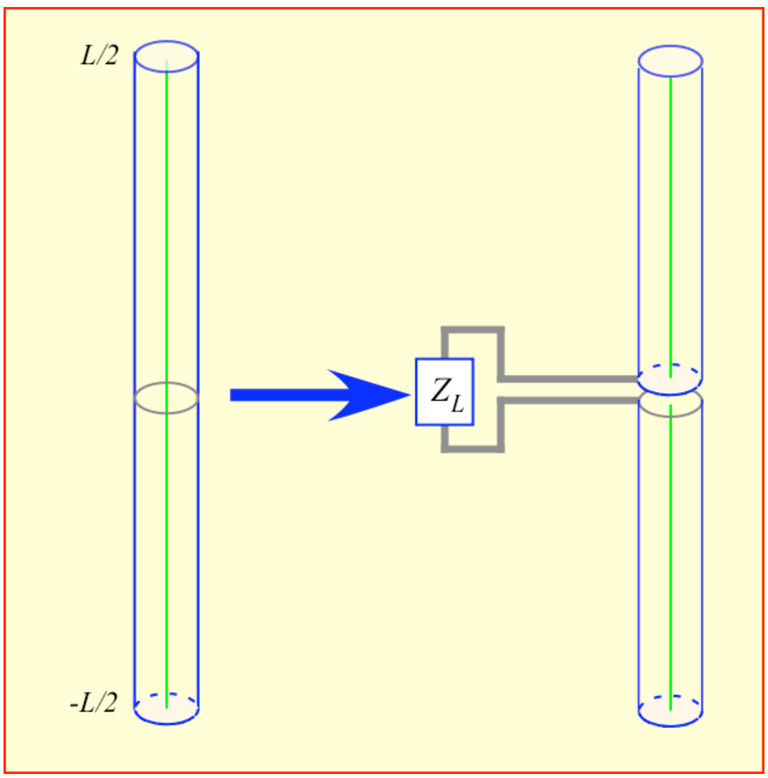

Figure 9: transmission line connected to the two halves of a wire to form a dipole

When the load impedance is 0, the ***short-circuit current*** flowing in the dipole is very similar to the current discussed above in the wire. When the load impedance is infinite, the ***open-circuit voltage*** is nearly equal to the voltage across a single wire half the length of the dipole, that is approximately $E_{inc}L/2$. The behavior of the antenna for any load can then be inferred from the open-circuit voltage and short-circuit current using Thévenin's theorem (figure 10). The current distribution for any load is the superimposition of the current distributions corresponding to the open-circuit and short-circuit conditions. The power delivered to a matched load is proportional to V_{oc}^2 /Re(Z_{eq}). At resonance the load is real and scales as the impedance of free space; thus the power delivered to the load has the same form as the scattered power (equation 1.19), and it is reasonable to infer that the two quantities are of similar magnitude: at resonance the antenna collects power from an ***effective area*** roughly equivalent to a square *L* on a side. For longer wavelengths or lower frequencies, the real part of the impedance falls as the square of the ratio of length to wavelength (equation 1.15), but the open-circuit voltage also falls linearly, so the power delivered to a matched load is ***independent of the size of the antenna***. (Note that in practice the load required becomes difficult to fabricate for very short antennas.)

A more precise treatment must incorporate the interaction of the two halves of the dipole, roughly a bit of capacitance between the ends and a small mutual inductance.

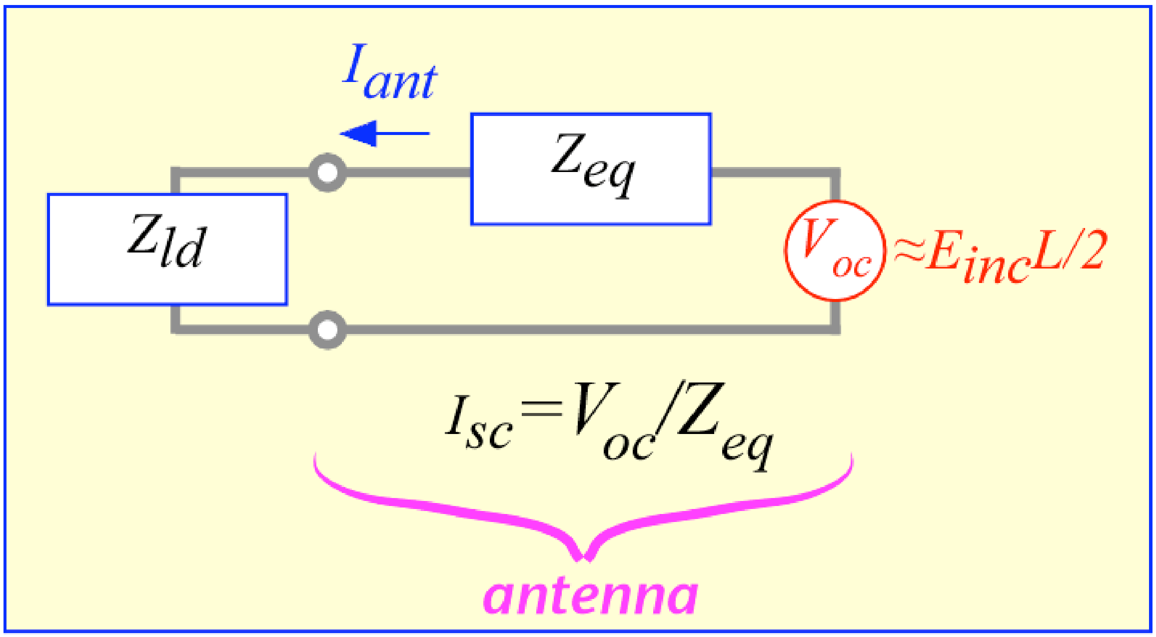

Figure 10: Thevenin equivalent circuit for dipole antenna; short-circuit current is I_0 of a wire of length L; open-circuit voltage is the voltage across a wire of length L/2.

To summarize the discussion so far, we have shown by heuristic arguments that current flow in a short wire ought to lead the incident voltage by 90 degrees (like a capacitive load), but that at some frequency where the wire is roughly a wavelength long, the current should be large and in phase with the incident voltage. At higher frequencies still, the current lags the voltage, as in an inductor. The portion of the current in phase with the incident voltage corresponds to power scattered away from the wire, mainly in the plane perpendicular to the wire axis. By splitting the wire at its center we obtain a dipole antenna, whose behavior as a receiver can be very simply estimated using the results we obtained for a continuous wire. As promised, all these results have been obtained without the need to find the magnetic field **B**.

It is important to note that these results apply only to antennas of total length comparable to or less than a wavelength; for longer antennas it is no longer tenable to approximate the phase as a linear function of distance, and contributions to the 'instantaneous' potentials from distant parts of the antenna change sign.

In part II of this article we will exhibit the detailed mathematics for implementing the calculations sketched out in part I, focusing on a simple quadratic approximation to the current distribution on the wire.

References

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4] **Collective Electrodynamics**, Mead, MIT Press 2002
5] **RF Engineering for Wireless Networks**, Dobkin, Elsevier 2004, Appendix 4