# Orientations and Rotations

## Introduction

The fact that accelerometers are sensitive to the gravitational force on the device allows them to be used to determine the attitude of the sensor with respect to the reference gravitational vector. This attitude determination is very useful in leveling or gimballing gyroscopes and magnetometers for use in compass and navigation instruments; determining tilt for game controller applications; and determining tilt or rotation for screen rotation of handheld devices. The method for calculating orientation or rotation depends on the specific application. In this application note a short introduction is given for some of the most common methods.

## Orientations

There are several methods for defining the orientation of an object. All involve describing the direction of a reference vector in a reference coordinate system. For accelerometers it is natural to use the response of the accelerometer to the static gravitational force as the reference vector, and in this case the reference coordinate system is the Earth with the positive z-axis pointed away from the center of the Earth.

The reference vector can be defined in a Cartesian sense by giving the components of the reference vector along the coordinate axes. If an accelerometer is not accelerating, then the three outputs of the accelerometer give the Cartesian components directly.

Another common method for defining this direction is in terms of direction cosines. In this case the angles that the reference vector makes with the coordinate axis are used. The equations below define how to determine the direction cosines from the accelerometer outputs.

$$\cos\alpha = \frac{a_x}{\sqrt{a_x^2 + a_y^2 + a_z^2}}$$

$$\cos\beta = \frac{a_y}{\sqrt{a_x^2 + a_y^2 + a_z^2}}$$

$$\cos\gamma = \frac{a_z}{\sqrt{a_x^2 + a_y^2 + a_z^2}}$$

In tilt applications such as game controllers you often are only concerned with the angle that the device has tilted away from the horizontal plane. In this case the direction sines would be used.

$$\sin\theta = \frac{\sqrt{a_y^2 + a_z^2}}{\sqrt{a_x^2 + a_y^2 + a_z^2}}$$

$$\sin\phi = \frac{\sqrt{a_x^2 + a_z^2}}{\sqrt{a_x^2 + a_y^2 + a_z^2}}$$

## Rotations

A general rotation, **A**, is typically defined in terms of a set of 3 rotations which can be represented as the product of successive matrices.

$$\mathbf{A} = \mathbf{BCD}$$

The specifics of the individual rotation matrices, unfortunately, are dependent on your field of study. Here we will discuss tow of the most common formulations; the "x-convention" typically used in mechanics; and the "xyz-convention" (yaw, pitch, roll) used in aeronautics.

### *"X-convention"*

The "x" convention used in mechanics the rotation is given by the Euler angles ($\varphi,\theta,\psi$) where the first angle, $\varphi$, is a rotation about the z-axis, the second angle, $\theta$, is a rotation about the old x-axis, and the third angle, $\psi$, is a rotation around the new z-axis. The component rotations are given as the following.

$$\mathbf{B} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So **A** is finally given as:

$$\mathbf{A}=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{aligned}
a_{11} &= \cos\psi\cos\varphi - \cos\theta\sin\psi \\
a_{12} &= \cos\psi\sin\varphi + \cos\theta\sin\psi \\
a_{13} &= \sin\psi\sin\theta \\
a_{21} &= -\sin\psi\cos\varphi - \cos\theta\sin\varphi\cos\psi \\
a_{22} &= -\sin\psi\sin\varphi + \cos\theta\cos\varphi\cos\psi \\
a_{23} &= \cos\psi\sin\theta \\
a_{31} &= \sin\theta\sin\varphi \\
a_{32} &= -\sin\theta\cos\phi \\
a_{33} &= \cos\theta
\end{aligned}$$

### *"XYZ-convention"*

The "xyz-convention" used in aeronautics is given by the Euler angles ($\varphi$ –roll, $\theta$ – pitch, $\psi$ – yaw) where the first angle, $\psi$, is a rotation about the z-axis, the second angle, $\theta$, is a rotation about the y-axis, and the third angle, $\varphi$, is a rotation around the x-axis. The component rotations are given as the following.

$$\mathbf{B}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & \sin\psi \\ 0 & -\sin\psi & \cos\psi \end{bmatrix}$$

$$\mathbf{C}=\begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$\mathbf{D}=\begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So **A** is finally given as:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$a_{11} = \cos\theta\cos\varphi$$
$$a_{12} = \cos\theta\sin\varphi$$
$$a_{13} = -\sin\theta$$
$$a_{21} = \sin\psi\sin\theta\cos\varphi - \cos\psi\sin\varphi$$
$$a_{22} = \sin\psi\sin\theta\sin\varphi + \cos\psi\cos\varphi$$
$$a_{23} = \cos\theta\sin\psi$$
$$a_{31} = \cos\psi\sin\theta\cos\varphi + \sin\psi\sin\varphi$$
$$a_{32} = \cos\psi\sin\theta\sin\varphi - \sin\psi\cos\varphi$$
$$a_{33} = \cos\theta\cos\psi$$

Both the “x convention” and the “xyz convention” suffer from a condition known as gimbal lock. In gimbal lock one of the rotations becomes large enough that 2 rotation axes become coincident and you lose a degree of freedom in your measurements. As an example imagine an airplane working in the “xyz convention, (pitch, roll and yaw). If the airplane first pitches up 90° now the roll axis and the yaw axis have become coincident. A common way of overcoming this issue is to use quaternions. Specifically for rotations these quaternions are sometimes called Euler parameters.

### *Quaternions*

Quaternions use 4 parameters to describe a rotation in 3 dimensions. Adding fourth parameter allows for avoiding the condition of gimbal lock. A simple way of looking at this is to first consider coordinates on a sphere. Using 2 parameters (latitude and longitude for example) we can determine the location on the sphere. However, at the North and South pole the two parameters become degenerate. If we add a third parameter and define the North pole as (+1,0,0), the South pole as (-1,0,0), and points on the equator (0,X,Y). This method can be extended to 3 dimensional rotations. The fourth parameter removes the degenerate rotations given by the Euler angles.

A quaternion can be described as a 4-parameter vector, e, as follows:

$$e = [e_0\ e_1\ e_2\ e_3]^{\mathrm{T}}$$
$$e_0^{\ 2} + e_1^{\ 2} + e_2^{\ 2} + e_3^{\ 2} \equiv 1$$

Sometimes it is helpful to think of the quaternion as a rotation, $\alpha$, around a direction axis defined by the direction cosines. The rotation matrix, **A**, for a unit quaternion is given by the following equation:

$$\begin{bmatrix} 1-2\left(e_2^2+e_3^2\right) & 2\left(e_1e_2-e_0e_3\right) & 2\left(e_0e_2-e_1e_3\right) \\ 2\left(e_1e_2+e_0e_3\right) & 1-2\left(e_1^2+e_3^2\right) & 2\left(e_2e_3-e_0e_1\right) \\ 2\left(e_1e_3-e_0e_2\right) & 2\left(e_0e_1+e_2e_3\right) & 1-2\left(e_1^2+e_2^2\right) \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$a_{11} = 1-2\left(e_2^2+e_3^2\right)$$
$$a_{12} = 2\left(e_1e_2-e_0e_3\right)$$
$$a_{13} = 2\left(e_0e_2-e_1e_3\right)$$
$$a_{21} = 2\left(e_1e_2+e_0e_3\right)$$
$$a_{22} = 1-2\left(e_1^2+e_3^2\right)$$
$$a_{23} = 2\left(e_2e_3-e_0e_1\right)$$
$$a_{31} = 2\left(e_1e_3-e_0e_2\right)$$
$$a_{32} = 2\left(e_0e_1+e_2e_3\right)$$
$$a_{33} = 1-2\left(e_1^2+e_2^2\right)$$

Comparing terms from this matrix with the rotation matrix in the “xyz-convention” we can determine the conversion equations between quaternions and the “xyz-convention.”

$$e_0 = \cos\frac{\varphi}{2}\cos\frac{\theta}{2}\cos\frac{\psi}{2} + \sin\frac{\varphi}{2}\sin\frac{\theta}{2}\sin\frac{\psi}{2}$$
$$e_1 = \sin\frac{\varphi}{2}\cos\frac{\theta}{2}\cos\frac{\psi}{2} - \cos\frac{\varphi}{2}\sin\frac{\theta}{2}\sin\frac{\psi}{2}$$
$$e_2 = \cos\frac{\varphi}{2}\sin\frac{\theta}{2}\cos\frac{\psi}{2} + \sin\frac{\varphi}{2}\cos\frac{\theta}{2}\sin\frac{\psi}{2}$$
$$e_3 = \cos\frac{\varphi}{2}\cos\frac{\theta}{2}\sin\frac{\psi}{2} - \sin\frac{\varphi}{2}\sin\frac{\theta}{2}\cos\frac{\psi}{2}$$

$$\varphi = \arctan\frac{2\left(e_0e_1 + e_2e_3\right)}{1-2\left(e_1^2 + e_2^2\right)}$$
$$\theta = \arcsin\left(2\left(e_0e_2 - e_1e_3\right)\right)$$
$$\psi = \arctan\frac{2\left(e_0e_3 + e_1e_2\right)}{1-2\left(e_2^2 + e_3^2\right)}$$

## Summary

Brief descriptions of orientation determination from accelerometer outputs were presented. Additionally, common methods for describing rotations were reviewed. Further information on the rotations can be found in most Classical Mechanics textbooks. There are also many helpful websites with articles that go into the methods in more depth.

## The Kionix Advantage

Kionix technology provides for X, Y, and Z-axis sensing on a single, silicon chip. One accelerometer can be used to enable a variety of simultaneous features including, but not limited to:

Hard Disk Drive protection
Vibration analysis
Tilt screen navigation
Sports modeling
Theft, man-down, accident alarm
Image stability, screen orientation & scrolling
Computer pointer
Navigation, mapping
Game playing
Automatic sleep mode

## Theory of Operation

Kionix MEMS linear tri-axis accelerometers function on the principle of differential capacitance. Acceleration causes displacement of a silicon structure resulting in a change in capacitance. A signal-conditioning CMOS technology ASIC detects and transforms changes in capacitance into an analog output voltage, which is proportional to acceleration. These outputs can then be sent to a micro-controller for integration into various applications. For product summaries, specifications, and schematics, please refer to the Kionix MEMS accelerometer product sheets at http://www.kionix.com/sensors/accelerometer-products.html.